

THE BOREL CARDINALITY OF LASCAR STRONG TYPES

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ABSTRACT. We show that if the restriction of the Lascar equivalence relation to a KP-strong type is non-trivial, then it is non-smooth (when viewed as a Borel equivalence relation on an appropriate space of types).

1. INTRODUCTION

Notions of strong type play an important role in the study of first-order theories. A *strong type* (over \emptyset) is a class of an automorphism-invariant equivalence relation on \mathfrak{C}^α which is bounded (i.e., the quotient has small cardinality) and refines equality of types. The phrase “strong type” by itself often refers to a *Shelah strong type*, which is simply a type over the algebraic closure of \emptyset (in T^{eq}). In other words, two sequences have the same Shelah strong type if they are equivalent with respect to every definable equivalence relation with finitely many classes. Refining this is the notion of *KP strong type* (\equiv_{KP}^α), in which two sequences are equivalent if they are equivalent with respect to every bounded type-definable equivalence relation. The KP strong type can also be characterized as the finest notion of strong type for which the corresponding quotient is a compact Hausdorff space. Finally, the *Lascar strong type* (\equiv_L^α) is the finest notion of strong type. The classes of \equiv_L^α coincide with the connected components of the *Lascar graph* on \mathfrak{C}^α , in which two sequences are neighbors if they lie along an infinite indiscernible sequence. The *Lascar distance* d is the associated graph distance. All of this is explained in detail in Subsection 1.1.

In [New03], Newelski established the following fundamental facts:

Fact 1.1. [New03] *Suppose that T is a complete first-order theory and α is an ordinal.*

- (1) *A Lascar strong type is type definable iff it has finite diameter.*
- (2) *If Y is an \equiv_L^α -invariant closed set, contained in some $p \in S_\alpha(\emptyset)$, on which every \equiv_L^α -class has infinite diameter, then Y contains at least 2^{\aleph_0} -many \equiv_L^α -classes.*
- (3) *Lascar strong types of unbounded diameter are not G_δ sets (when viewed as subsets of an appropriate space of types, as explained in Subsection 1.4).*
- (4) *If T is small (i.e., T is countable and the number of finitary types over \emptyset is countable), then $\equiv_L^n = \equiv_{KP}^n$ for all $n < \omega$ (i.e., the two notions of type agree on finite sequences).*

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As opposed to Shelah and KP strong types, the space of Lascar strong types does not come equipped with a Hausdorff topology. It is therefore unclear to what category this quotient belongs. In [KPS12], the authors suggest viewing it through the framework of descriptive set theory (this idea was already mentioned in [CLPZ01]). They formally interpret the notion of equality of Lascar-strong types as a Borel equivalence relation over a compact Polish space, and then consider the position of this relation in the Borel reducibility hierarchy.

Given two Polish spaces X and X' and two Borel equivalence relations E and E' respectively on X and X' , we say that E is *Borel reducible* to E' if there is a Borel map f from X to X' such that $x E y \iff f(x) E' f(y)$ for all $x, y \in X$. Two relations are *Borel bi-reducible* if each is Borel reducible to the other. The quasi-order of Borel reducibility is a well-studied object in descriptive set theory, and enjoys a number of remarkable properties. One of them is given by the Harrington-Kechris-Louveau dichotomy, which asserts that a Borel equivalence relation is either smooth (Borel reducible to equality on 2^ω) or at least as complicated as \mathbb{E}_0 (eventual equality on 2^ω). This is explained in detail in Subsections 1.2 and 1.2.

In this paper, we provide the following solution to the main conjecture of [KPS12].

Main Theorem A. *[Simplified version] Suppose that T is a complete countable first-order theory. If \equiv_L does not coincide with \equiv_{KP} , then \equiv_L is not smooth.*

We will actually prove a slightly stronger result (see Theorem 4.12). Our proof will not make use of Fact 1.1, which we will recover (for countable T) as a corollary.

Let us say a few words about our method. In [NP06], Newelski and Petrykowski introduce the notion of weakly generic types for definable groups. An analog for groups of automorphisms was used in Pelaez's thesis [Pel08] to give an alternate proof of Fact 1.1 (1). We follow this lead in our own proof.

A consequence of the proof of an early special case of the Harrington-Kechris-Louveau dichotomy theorem is that if X is a Polish space, G is a Polish group acting continuously on X , and the orbit equivalence relation E_G is F_σ , then either E_G is smooth or \mathbb{E}_0 can be continuously embedded into E_G . This is related to [BK96, Theorem 3.4.5], which gives a sufficient condition for embedding \mathbb{E}_0 in an equivalence relation induced by a group action (namely that there is a dense orbit and that E_G is meager).

As a corollary of the latter result, we establish a sufficient criterion for embedding \mathbb{E}_0 into an equivalence relation E whose classes are each equipped with a metric. Roughly speaking, the group G of homeomorphisms of the Cantor space whose graphs are contained in \mathbb{E}_0 acts in a sufficiently rich fashion that it can move any element to another which is arbitrarily close in the topological sense, but far away in the sense of the metric associated with the class. We show that if a similar property holds for E then one can embed \mathbb{E}_0 into E .

Assuming that there is a Lascar-strong type of unbounded diameter, it is thus natural to try to find a type p whose orbit under the group of Lascar-strong automorphisms is also sufficiently rich. When T is countable, we construct such a type formula-by-formula. At each stage, we must make sure that we still have room to go on, namely, that the partial type has many images which are at large Lascar distance from each other. To this end, we make sure that the type stays weakly generic. We actually need a slightly stronger property which we call “properness”.

Main Theorem A does not seem to be enough to deduce Fact 1.1 for uncountable theories. However, we adapt our argument to also take care of this case. This is Main Theorem B. For uncountable languages, the space of types is no longer a Polish space, so we do not state the theorem in terms of Borel cardinality. Apart from that, the result is essentially the same as for countable theories. In particular, it implies Newelski’s theorem. In order to prove it, we will need a little bit more from the descriptive set theoretic side, namely, the notion of (strong) Choquet space. This is used to replace completeness. In fact, eventually we deal with a non-Hausdorff space. (A non-Hausdorff space will arise as the space of types over a model M with the topology induced by a countable sub-language L' of L and a corresponding countable model M' .)

Organization of the paper. We have made an effort to keep this paper self-contained and accessible to model theorists and descriptive set theorists alike. Thus we start by giving all of the required definitions from both sides. In Section 2, we state a set theoretic criterion for non-smoothness. In Section 3, we treat a special case of the main theorem, where T is small (hence Lascar strong types coincide with KP strong types on finite tuples) and α is infinite. Although this result will not be used to prove the general case, we thought it worthwhile to include, as the proof is considerably simpler and gives insight into the general case. In Section 4, we prove Main Theorem A for all countable theories. Finally, in section 5, we prove Main Theorem B, thereby taking care of the general case.

1.1. Model-theoretic preliminaries. Let T be any complete first order theory. The theory T may be many sorted, but for the simplicity of the presentation one may assume that it is one sorted. We recall some basic notions. We fix a sequence of variables $\langle v_i \mid i \in \mathbf{ord} \rangle$. For the rest of this section, α will be some ordinal.

Definition 1.2. Suppose $M \models T$, $A \subseteq M$. Let $L_\alpha(A)$ the set of formulas in variables $\langle v_i \mid i < \alpha \rangle$ with parameters from A — the set of *formulas over A* . An α -*type over A* (sometimes a *partial α -type over A*) is a consistent subset of $L_\alpha(A)$. An α -type p over A is called *complete* if for any formula $\varphi \in L_\alpha(A)$, $\{\varphi, \neg\varphi\}$ intersects p . We denote by $S_\alpha(A)$ the space of all complete α -types over A . For a tuple $a \in M^\alpha$, let

$$\mathrm{tp}(a/A) = \{\varphi \in L_\alpha(A) \mid M \models \varphi(a)\}.$$

We write $a \equiv_A b$ for $\text{tp}(a/A) = \text{tp}(b/A)$. If $A = \emptyset$, we omit it. If p is an α -type over A and $\text{tp}(a/A) \supseteq p$ we write $a \models p$. We say that p is *realized* in M if there exists some $a \in M^\alpha$ such that $a \models p$.

We sometimes write $p(x)$ (respectively $\varphi(x)$) when we want to stress that the free variables of p (respectively φ) are contained in the tuple of variables x .

Remark 1.3. The set $S_\alpha(A)$ is naturally a compact Hausdorff topological space, the *Stone space* of α -types over A , with clopen basis $\{[\varphi] \mid \varphi \in L_\alpha(A)\}$, where $[\varphi] = \{p \in S_\alpha(A) \mid \varphi \in p\}$.

Definition 1.4. For a cardinal κ , a model $M \models T$ is called κ -*saturated* when for all sets $A \subseteq M$ of cardinality $< \kappa$ all types in $S_1(A)$ are realized in M . The model M is κ -*homogeneous* when for all $\alpha < \kappa$ and $a, b \in M^\alpha$ if $a \equiv b$ then there is an automorphism of M mapping a to b .

Recall that the cardinality of T , $|T|$, is identified with the cardinality of the set of formulas in T .

Fact 1.5. [Hod93, Theorem 10.2.1] *For any cardinal $\kappa \geq |T|$, and any model $M \models T$ there exists a κ -saturated, κ -homogeneous model $N \succ M$ of T .*

Fix some $\kappa \geq (2^{|T|})^+$. We denote by \mathfrak{C} a κ -saturated, κ -homogeneous model of T . In model theory, this is usually referred to as the “monster model” of T (and it is often harmless to assume in addition that \mathfrak{C} is $|\mathfrak{C}|$ -saturated and is very big). The convention is that all sets of parameters and tuples we deal with are small, that is, of cardinality $< \kappa$, and that they are all contained in \mathfrak{C} . Similarly all small models are assumed to be elementary substructures of \mathfrak{C} .

Recall that for $A \subseteq \mathfrak{C}$, $\text{Aut}(\mathfrak{C}/A)$ denotes the group of automorphisms of \mathfrak{C} which fix A pointwise.

Definition 1.6. Let $A \subseteq \mathfrak{C}$ a small set. A set $X \subseteq \mathfrak{C}^\alpha$ is called *A-type-definable* (or *type-definable over A*) if it is empty or there is an α -type p over A such that

$$X = \{a \in \mathfrak{C}^\alpha \mid a \models p\}.$$

It is *A-invariant* (or *invariant over A*) when for all $\sigma \in \text{Aut}(\mathfrak{C}/A)$, $\sigma^\alpha(X) = X$ (usually we omit α from this notation). When A is omitted, it is understood that $A = \emptyset$.

We define a “topology” on subsets of \mathfrak{C}^α .

Definition 1.7. Call a subset $X \subseteq \mathfrak{C}^\alpha$ *pseudo closed* if X is type definable over some small set. A *pseudo open* set is a complement of a pseudo closed set. *Pseudo G_δ* sets and *pseudo F_σ* sets are defined in the obvious way.

By saturation \mathfrak{C}^α is pseudo compact in the sense that any small intersection of non-empty pseudo closed sets is non-empty. This is why we often say “by compactness”, instead of “by saturation”.

Remark 1.8. By compactness, for a small set $A \subseteq \mathfrak{C}$, the map $r_{\alpha,A} : \mathfrak{C}^\alpha \rightarrow S_\alpha(A)$ defined by $a \mapsto \text{tp}(a/A)$ is pseudo closed, in the sense that it sends pseudo closed sets to closed sets (in the Stone topology). So $r_{\alpha,A}$ maps pseudo F_σ subsets of \mathfrak{C}^α to F_σ subsets of $S_\alpha(A)$.

We also recall the notion of an indiscernible sequence:

Definition 1.9. Let A be a small set. Let $(I, <_I)$ be some linearly ordered set. A sequence $\bar{a} = \langle a_i \mid i \in I \rangle \in (\mathfrak{C}^\alpha)^I$ is called *A-indiscernible* (or *indiscernible over A*) if for all $n < \omega$, every increasing n -tuple from \bar{a} realizes the same type over A . When A is omitted, it is understood that $A = \emptyset$.

An easy but very important fact about indiscernible sequences is that they exist.

Fact 1.10.

- (1) [TZ12, Lemma 5.1.3] *Let $(I, <_I)$, $(J, <_J)$ be small linearly ordered sets, and let A be some small set. Suppose $\bar{b} = \langle b_j \mid j \in J \rangle$ is some sequence of elements from \mathfrak{C}^α . Then there exists an indiscernible sequence $\bar{a} = \langle a_i \mid i \in I \rangle \in (\mathfrak{C}^\alpha)^I$ such that:*
 - *For any $n < \omega$ and $\varphi \in L_{\alpha,n}$, if $\mathfrak{C} \models \varphi(b_{j_0}, \dots, b_{j_{n-1}})$ for every $j_0 <_J \dots <_J j_{n-1}$ from J then $\mathfrak{C} \models \varphi(a_{i_0}, \dots, a_{i_{n-1}})$ for every $i_0 <_I \dots <_I i_{n-1}$ from I .*
- (2) [Ker07, proof of Proposition 3.1.4] *If M is a small model and $a \equiv_M b$, then there is an indiscernible sequence $\bar{c} = \langle c_i \mid i < \omega \rangle$ such that both $a \frown \bar{c}$ and $b \frown \bar{c}$ are indiscernible.*

Point (1) in Fact 1.10 is proved using Ramsey's theorem and compactness, while (2) is proved with ultrafilters.

Definition 1.11. An equivalence relation E on a set X is called *bounded* if $|X/E| < \kappa$.

Remark 1.12. By saturation and homogeneity, every invariant set is a union of types over \emptyset . So by saturation if E is an invariant equivalence relation with an invariant domain $X \subseteq \mathfrak{C}^\alpha$, it makes sense to consider E in any monster model. When E is bounded, and $|\alpha| \leq |T|$ then $|X/E| \leq 2^{|T|}$. To see that, let $M \models T$ be of size $|T|$. If $a, b \in X$ and $a \equiv_M b$ then $(a, b) \in E$, since otherwise, by Fact 1.10 (2) and saturation, we may assume that $\langle a, b \rangle$ starts an indiscernible sequence of length κ . By homogeneity, any two elements in it are not E -equivalent. Now the result follows from the fact that $|S_\alpha(M)| \leq 2^{|T|}$. It is now easy to see that if $\mathfrak{C}' \succ \mathfrak{C}$ is another monster model then every E -class in \mathfrak{C}' intersects \mathfrak{C} , so there are no “new” classes.

We come to the central definition.

Definition 1.13. The *Lascar graph* on \mathfrak{C}^α is the set G_α of pairs (a, b) of distinct elements of \mathfrak{C}^α which lie along an infinite indiscernible sequence. The Lascar metric d_α is the metric associated with this graph. Let \equiv_L^α denote the equivalence relation on \mathfrak{C}^α whose classes coincide with the

connected components of G_α . The *Lascar strong type* of a tuple $a \in \mathfrak{C}^\alpha$ is its \equiv_L^α -class. We will omit α from the notation when it is clear from context.

Remark 1.14. By Fact 1.10 (2), it follows that for $a, b \in \mathfrak{C}^\alpha$ and $M \prec \mathfrak{C}$, if $a \equiv_M b$ then $d(a, b) \leq 2$.

Fact 1.15. (see e.g., [Ker07, Proposition 3.1.4]) *The relation \equiv_L^α is the finest bounded invariant equivalence relation on \mathfrak{C}^α .*

Proof (sketch.) If E is some bounded invariant equivalence relation on \mathfrak{C}^α and $d_\alpha(a, b) \leq 1$, then as in Remark 1.12, $(a, b) \in E$. Similarly, \equiv_L^α is bounded since it is bounded by $|S_\alpha(M)|$ for any model $M \models T$. \square

Definition 1.16. The group of *Lascar strong automorphisms* of \mathfrak{C} is the group generated by automorphisms σ of \mathfrak{C} for which there is a small model $M \prec \mathfrak{C}$ fixed pointwise by σ , i.e., the group

$$\text{Aut } f_L(\mathfrak{C}) = \langle \sigma \in \text{Aut}(\mathfrak{C}/M) \mid M \prec \mathfrak{C} \rangle.$$

Fact 1.17. (see e.g., [Ker07, Section 3.1])

- (1) *The group of Lascar strong automorphisms is a normal subgroup of $\text{Aut}(\mathfrak{C})$. It consists of all automorphisms that fix all Lascar strong types (of any length).*
- (2) *The Lascar strong type equivalence relation is the orbit equivalence relation of the group of Lascar strong automorphisms.*
- (3) *If σ is a Lascar strong automorphism, then there is some $m < \omega$ such that for any tuple c (of any length), $d(c, \sigma(c)) \leq m$. In this case we say that m bounds σ .*

Remark 1.18. Suppose $a, b \in \mathfrak{C}^\alpha$ and $d_\alpha(a, b) \leq n$. Then there is a Lascar strong automorphism σ of \mathfrak{C} bounded by $2n$ such that $\sigma(a) = b$.

Proof. (of Remark 1.18) It is enough to establish it in the case $d(a, b) \leq 1$: if $d(a, b) \leq n$, then there are c_0, \dots, c_n with $a = c_0$, $c_n = b$ and $d(c_i, c_{i+1}) \leq 1$ for all $i < n$. For each $i < n$, we find some σ_i bounded by 2 that maps c_i to c_{i+1} . Let $\sigma = \sigma_{n-1} \circ \dots \circ \sigma_0$.

So suppose $I = \langle a_i \mid i < \omega \rangle$ is an indiscernible sequence that starts with $a_0 = a, a_1 = b$. Let M be a model of size $|T|$. By saturation we can extend the sequence I to length $(2^{|T|})^+$. So there must be two elements in I that have the same type over M . By indiscernibility and homogeneity, there is some model M' such that $a \equiv_{M'} b$.

The remark now follows from Remark 1.14. \square

We also recall the notion of KP strong type:

Definition 1.19. Let \equiv_{KP}^α denote the finest bounded type-definable equivalence relation on \mathfrak{C}^α . The *KP strong type*¹ of a tuple $a \in \mathfrak{C}^\alpha$ is its \equiv_{KP}^α -class.

¹KP stands for Kim-Pillay. This notation was introduced by Hrushovski in [Hru98].

Fact 1.20. [Cas11, Proposition 15.25] *Let X be any type-definable subset of \mathfrak{C}^α .*

- (1) *The restriction $\equiv_L^\alpha \upharpoonright X$ of \equiv_L^α to X is the finest bounded invariant equivalence relation on realizations of X .*
- (2) *The restriction $\equiv_{KP}^\alpha \upharpoonright X$ of \equiv_{KP}^α to X is the finest bounded type-definable equivalence relation on realizations of p .*

Remark 1.21. By saturation and homogeneity if $X \subseteq \mathfrak{C}^\alpha$ is type-definable over some small set B and invariant over another small set A , then it is type-definable over A . It follows that if $K \subseteq \mathfrak{C}^\alpha$ is a KP strong type, and for some $a \in K$, $[a]_{\equiv_L^\alpha}$ is pseudo closed, then $\equiv_L^\alpha \upharpoonright K$ is trivial. Indeed, it is type-definable over a so there is a type $\pi(x, y)$ such that $\pi(x, a)$ defines $[a]_{\equiv_L^\alpha}$. Let $p(x) = \text{tp}(a/\emptyset)$. Then $\equiv_L^\alpha \upharpoonright p$ is defined by: $x \equiv_L^\alpha y$ iff $\pi(x, y)$. Fact 1.20 implies that $\equiv_L^\alpha \upharpoonright p = \equiv_{KP}^\alpha \upharpoonright p$, so $\equiv_L^\alpha \upharpoonright K$ is trivial.

Definition 1.22. Let $Y \subseteq \mathfrak{C}^\alpha$ be closed under \equiv_L^α . We say that Y is *d-bounded* if there is some $n < \omega$ such that $a \equiv_L^\alpha b$ iff $d(a, b) \leq n$ for all $a, b \in Y$.

Remark 1.23. For a set of parameters A , *Lascar distance over A* , *Lascar strong type over A* , *KP-strong type over A* , etc., are the parallel notions for T_A : the complete theory of the structure \mathfrak{C}_A which is just \mathfrak{C} after naming all elements from A . All the facts above hold for A with the obvious adjustments.

1.2. Preliminaries on Borel equivalence relations. Here we give the basic facts about Borel equivalence relations.

Definition 1.24. Suppose X and Y are Polish spaces, and E and F are Borel equivalence relations on X and Y . We say that a function $f : X \rightarrow Y$ is a *reduction* of E to F if for all $x_0, x_1 \in X$, $(x_0, x_1) \in E$ iff $(f(x_0), f(x_1)) \in F$.

- (1) We say that E is *Borel reducible* to F , denoted by $E \leq_B F$, when there is a Borel reduction $f : X \rightarrow Y$ of E to F .
- (2) We say that E is *continuously reducible* to F , denoted by $E \sqsubseteq_c F$, when there is a continuous injective reduction $f : X \rightarrow Y$ of E to F .
- (3) We say that E and F are *Borel bi-reducible*, denoted by $E \sim_B F$, when $E \leq_B F$ and $F \leq_B E$.
- (4) We write $E <_B F$ to mean that $E \leq_B F$ but $E \not\sim_B F$.

Example 1.25. For a Polish space X , the relations $\Delta(X)$ denotes equality on X . Then $\Delta(1) <_B \Delta(2) <_B \dots <_B \Delta(\omega) <_B \Delta(2^\omega)$.

Definition 1.26. We say that E is *smooth* iff $E \leq_B \Delta(2^\omega)$.

Note that being smooth is equivalent to the existence of “separating Borel sets,” i.e., Borel sets $B_i \subseteq X$ such that $x E y$ iff for all $i < \omega$, $x \in B_i$ iff $y \in B_i$.

Fact 1.27. [Sil80] (*Silver dichotomy*) For all Borel equivalence relations E , $E \leq_B \Delta(\omega)$ or $\Delta(2^\omega) \subseteq_c E$. It follows that $\Delta(2^\omega)$ is the successor of $\Delta(\omega)$.

Proposition 1.28. *Closed equivalence relations are smooth.*

Proof. Suppose E is a closed equivalence relation on a Polish space X . We must find Borel set $B_i \subseteq X$ for $i < \omega$ such that $x E y$ iff for all $i < \omega$, $x \in B_i \Leftrightarrow y \in B_i$. Since $X^2 \setminus E$ is open, it equals $\bigcup_{i < \omega} U_i \times V_i$ for $U_i, V_i \subseteq X$ open. Let $U_i^E = \{x \in X \mid \exists y (y \in U_i \& x E y)\}$ be the E -closure of U_i and V_i^E be the E -closure of V_i . These are analytic sets. Since $U_i^E \cap V_i^E = \emptyset$, by Lusin’s separation theorem, there are Borel sets U_i^0 such that $U_i^0 \supseteq U_i^E$, $U_i^0 \cap V_i^E = \emptyset$. Recursively we construct Borel sets U_i^j for $j < \omega$ such that U_i^j contains the E -closure of U_i^{j-1} and is disjoint from V_i^E . Let $B_i = \bigcup_{j < \omega} U_i^j$. \square

Example 1.29. Let \mathbb{E}_0 be the following equivalence relation on the Cantor space 2^ω : $(\eta, \nu) \in \mathbb{E}_0$ iff there exists some $n < \omega$ such that for all $m > n$, $\eta(m) = \nu(m)$.

Proposition 1.30. *The relation \mathbb{E}_0 is non-smooth.*

Proof. Recall that all Borel subsets B of a Polish space X have the Baire property: there is an open set $O \subseteq X$ such that $O \Delta B$ is meager. Suppose $\{B_i \mid i \in \omega\}$ are Borel separating sets of \mathbb{E}_0 , so all of them have the Baire property.

Fix some $i < \omega$, and suppose B_i is not meager. Then there is some $n < \omega$ and some $s \in 2^n$ such that, letting $O_s = \{\eta \in 2^\omega \mid s \triangleleft \eta\}$, $O_s \setminus B_i$ is meager. Let $t \in 2^n$. Since B_i is closed under \mathbb{E}_0 , and there is a homeomorphism of 2^ω taking O_s to O_t fixing all \mathbb{E}_0 -classes, $O_t \setminus B_i$ is also meager. But then $2^\omega \setminus B_i = \bigcup_{s \in 2^n} O_s \setminus B_i$ is meager, so B_i is comeager. This shows that B_i is either meager or comeager.

But then,

$$B = \bigcap \{B_i \mid i < \omega, B_i \text{ is comeager}\} \cap \bigcap \{\sim B_i \mid i < \omega, B_i \text{ is meager}\}$$

is a comeager \mathbb{E}_0 -class, which is a contradiction (since it is countable). \square

In addition, we have the following dichotomy:

Fact 1.31. [HKL90] (*Harrington-Kechris-Louveau dichotomy*) For every Borel equivalence relation E either $E \leq_B \Delta(2^\omega)$ (i.e., E is smooth) or $\mathbb{E}_0 \subseteq_c E$. It follows that \mathbb{E}_0 is the successor of $\Delta(2^\omega)$.

We also mention:

Corollary 1.32. *Suppose Y is a Polish space, and E is a Borel equivalence relation on Y such that all its classes are G_δ -subsets. Then E is smooth.*

Proof. Suppose E is not smooth. By Fact 1.31, there is a continuous map $f : 2^\omega \rightarrow Y$ that reduces \mathbb{E}_0 to E . But then it follows that all the \mathbb{E}_0 -classes are continuous pre-images of G_δ sets, so they are themselves G_δ . As they are also dense, this is a contradiction. \square

1.3. Preliminaries on Choquet spaces. As we mentioned above, when the language is not necessarily countable we will work with Choquet spaces instead of Polish spaces.

Definition 1.33. The *Choquet game* on a topological space X is a two player game in ω -many rounds. In round n , player A chooses a non-empty open set $U_n \subseteq V_{n-1}$ (where $V_{-1} = X$), and player B responds by choosing a non-empty open subset $V_n \subseteq U_n$. Player B wins if the intersection $\bigcap \{V_n \mid n < \omega\}$ is not empty.

The *strong Choquet game* is similar: in round n player A chooses an open set $U_n \subseteq V_{n-1}$ and $x_n \in U_n$, and player B responds by choosing an open set $V_n \subseteq U_n$ containing x_n . Again, player B wins when the intersection $\bigcap \{V_n \mid n < \omega\}$ is not empty.

A topological space X is a *(strong) Choquet space* if player B has a winning strategy in every (strong) Choquet game.

Given a subset A of X , we say that X is *strong Choquet over A* to mean that the points that player A chooses are taken from A .

It is easy to see that:

Example 1.34. Every Polish space is strong Choquet.

But for our purposes, we shall need the following example:

Example 1.35. If X is compact (not necessarily Hausdorff) and has a basis consisting of clopen sets then it is strong Choquet.

Proof. In round n , player B will choose a clopen set $x_n \in V_n \subseteq U_n$. By compactness, the intersection $\bigcap \{V_n \mid n < \omega\}$ is not empty. \square

Proposition 1.36. *If X is strong Choquet and $\emptyset \neq U \subseteq X$ is G_δ , then U is also strong Choquet.*

Proof. Suppose $U = \bigcap \{W_n \mid n < \omega\}$ where $W_n \subseteq X$ are open. Let St be a strategy for the strong Choquet game in X and we will describe a strategy St_U for the strong Choquet game in U . So we play a game \mathcal{D}_U in U , and we run a parallel game \mathcal{D}_X in X as follows. Assume we have already played all the rounds up to n : the sets U_i, V_i were chosen for $i < n$ in the game \mathcal{D}_U , and U'_i, V'_i are the corresponding moves in the \mathcal{D}_X . The construction will ensure that for all $i < n$, we have

$U'_i \cap U = U_i$, $V'_i \cap U = V_i$ and $U'_n \subseteq W_n$. Assume that A plays (U_n, x_n) , with $x_n \in U_n$. Pick an open subset U_* of X such that $U_* \cap U = U_n$. We set A's move in the parallel game to be $(U_* \cap W_n \cap V'_{n-1}, x_n)$. Let V'_n be B's move according to the strategy St . Then in \mathcal{D}_U , have B play $V'_n \cap U$. Note that this set is non-empty since it contains x_n . This defines a winning strategy for B. \square

1.4. Context.

1.4.1. *Countable language.* In [KPS12], the authors gave a natural way of considering \equiv_L^α and \equiv_{KP}^α for a countable complete first order theory T and a countable α as Borel equivalence relations on the space of types $S_\alpha(M)$ over a countable model M (this is a Polish space — see Remark 1.3 about the topology). Fix some countable T and α .

Definition 1.37. Let M be a countable model. For $p, q \in S_\alpha(M)$, we write $p \equiv_L^{\alpha, M} q$ iff $\exists a \models p, b \models q$ ($a \equiv_L^\alpha b$) and similarly we define $\equiv_{KP}^{\alpha, M}$.

It will be useful to define the Lascar metric on types:

Definition 1.38. For $p, q \in S_\alpha(M)$ let $d_\alpha(p, q) = \min \{n \in \mathbb{N} \mid \exists a \models p, b \models q \text{ } (d_\alpha(a, b) \leq n)\}$.

Note that:

Remark 1.39. [KPS12, Remark 2.2] Let M be a countable model. By Remark 1.14, for $p, q \in S_\alpha(M)$, $p \equiv_L^{\alpha, M} q$ iff $\forall a \models p, b \models q$ ($a \equiv_L^\alpha b$) and similarly for $\equiv_{KP}^{\alpha, M}$.

Let $q_{\alpha, M} : S_{\alpha \cdot 2}(M) \rightarrow S_\alpha(M)$ be defined by $p(x, y) \mapsto (p \upharpoonright x, q \upharpoonright y)$. This is a continuous map, and hence it is closed. Using this notation, $\equiv_{KP}^{\alpha, M} = q_{\alpha, M} \circ r_{\alpha \cdot 2, M}(\equiv_{KP}^\alpha)$ (see Remark 1.8), and hence $\equiv_{KP}^{\alpha, M}$ is closed. Similarly, the set

$$F_n = \{(p, q) \in S_\alpha(M) \mid d_\alpha(p, q) \leq n\}$$

is closed, and $\equiv_L^{\alpha, M}$ is the union $\bigcup_{n < \omega} F_n$ hence it is K_σ .

They proved that as far as Borel cardinality goes, this does not depend on the model M , even when restricting to a KP strong type:

Fact 1.40. [KPS12, Propositions 2.3, 2.6] *Let M and N be any countable models. Then,*

- (1) $\equiv_L^{\alpha, M} \sim_B \equiv_L^{\alpha, N}$.
- (2) For any $a \in \mathfrak{C}$, $\equiv_L^{\alpha, M} \upharpoonright [\text{tp}(a/M)]_{\equiv_{KP}^{\alpha, M}} \sim_B \equiv_L^{\alpha, N} \upharpoonright [\text{tp}(a/N)]_{\equiv_{KP}^{\alpha, N}}$.

One can extend this observation to deal also with pseudo G_δ sets. Suppose $Y \subseteq \mathfrak{C}^\alpha$ is a pseudo G_δ set. For a countable model M , $Y_M = r_{\alpha, M}(Y) \subseteq S_\alpha(M)$ is not necessarily G_δ . But in case Y is closed under \equiv_L^α , it is. Indeed, $\mathfrak{C}^\alpha \setminus Y$ is pseudo F_σ , and so $r_{\alpha, M}(\mathfrak{C}^\alpha \setminus Y)$ is F_σ . But by Remark 1.14, $r_{\alpha, M}(\mathfrak{C}^\alpha \setminus Y) \cap Y_M = \emptyset$.

For a countable model M , Y_M is a Polish space (as every G_δ set is). In addition, changing the model does not change the Borel cardinality:

Proposition 1.41. *Fix a pseudo G_δ set $Y \subseteq \mathfrak{C}^\alpha$, closed under \equiv_L^α . Then*

$$\equiv_L^{\alpha,M} \restriction Y_M \sim_B \equiv_L^{\alpha,N} \restriction Y_N.$$

Proof. The proof is exactly the same as in [KPS12, Propositions 2.3, 2.6], but we repeat it for completeness. It is enough to establish this when $M \subseteq N$. Let $\pi : S_\alpha(N) \rightarrow S_\alpha(M)$ be the restriction map. Then π is a continuous map that reduces $\equiv_L^{\alpha,N}$ to $\equiv_L^{\alpha,M}$. By [KPS12, Fact 1.7 (i)] there is a Borel section, i.e., a Borel function $\pi' : S_\alpha(M) \rightarrow S_\alpha(N)$ such that $\pi \circ \pi' = \text{id}$. Now it follows that π and π' restricted to Y_M and Y_N witness Borel bi-reducibility. \square

This allows us to refer to the Borel cardinality of $\equiv_L^\alpha \restriction Y$ without specifying the model.

1.4.2. Countable or uncountable language. Let T be any complete first order theory and α any ordinal. In order to state our theorem in full generality, we shall need the following definition:

Definition 1.42. We say that a set $Y \subseteq \mathfrak{C}^\alpha$ for some small α is *pseudo strong Choquet* if Y_M is strong Choquet for all M .

Example 1.43. Pseudo closed and pseudo G_δ sets which are closed under \equiv_L^α are pseudo strong Choquet by the observation after Fact 1.40 and Proposition 1.36.

Remark 1.44. For countable T and α , “pseudo strong Choquet” is the correct analog of pseudo G_δ for sets closed under \equiv_L^α . By [Kec95, Theorem 8.17] if $Y \subseteq \mathfrak{C}^\alpha$ is such a set, then Y is pseudo strong Choquet iff Y is pseudo G_δ iff Y_M is Polish for every M .

1.5. Results. Our main theorem, proved in Section 4, is:

Main Theorem A. *Suppose T is a complete countable first-order theory, α a countable ordinal, and suppose Y is a pseudo G_δ subset of \mathfrak{C}^α which is closed under \equiv_L^α . If for some $a \in Y$, $[a]_{\equiv_L^\alpha}$ is not d -bounded, then $\equiv_L^\alpha \restriction Y$ is non-smooth.*

Remark 1.45. This theorem remains true also for many-sorted countable theories, with the obvious adjustments.

We immediately get Conjecture 1 of [KPS12]:

Corollary 1.46. *Suppose T and α are as above. Suppose $K \subseteq \mathfrak{C}^\alpha$ is a KP strong type. If $\equiv_L^\alpha \restriction K$ is not d -bounded, then $\equiv_L^\alpha \restriction K$ is non-smooth. In particular, by Remark 1.21, if $\equiv_L^\alpha \restriction K$ is not trivial, then it is non-smooth.*

Proof. Observe that if $\equiv_L^\alpha \upharpoonright K$ is not d -bounded, then there is a \equiv_L^α -class inside K which is not d -bounded (else all classes will have the same bound, since they are conjugates). \square

Corollary 1.47. *Suppose T and α are as above. Then \equiv_L^α is closed iff it is smooth.*

Proof. If \equiv_L^α is not closed, then $\equiv_L^\alpha \neq \equiv_{KP}^\alpha$, so there is a KP strong type K such that $\equiv_L^\alpha \upharpoonright K$ is not trivial, so $\equiv_L^\alpha \upharpoonright K$ is not smooth, so also \equiv_L^α . The other direction follows from Proposition 1.28. \square

Remark 1.48. Since our main result concerns \mathbb{E}_0 , it actually says something about the “definable cardinality” of \equiv_L^α , i.e., it is stronger than just saying something about the Borel cardinality of \equiv_L^α , but also allows reductions to be “definable”. In the proof of Proposition 1.30, we showed that there are no separating sets for \mathbb{E}_0 with the Baire property. In any reasonable interpretation of the term, any “definable” reduction of \mathbb{E}_0 to $\Delta(Y)$ for some Polish space Y will give rise to such separating sets. So our main result implies that the “definable cardinality” of \equiv_L^α is greater than $\Delta(2^\omega)$. We will not give an exact definition of “definable cardinality” (see more in [BK96, Chapter 8]).

For a general language and α we have:

Main Theorem B. *[Simplified version] Suppose T is a complete first-order theory, α a small ordinal. Suppose $Y \subseteq \mathfrak{C}^\alpha$ is closed under \equiv_L^α and for some $a \in Y$, $[a]_{\equiv_L^\alpha}$ is not d -bounded. Suppose Y is pseudo strong Choquet. Then $|Y/\equiv_L^\alpha| \geq 2^{\aleph_0}$.*

The full theorem says a bit more, see 5.1.

Corollary 1.49. *Fact 1.1 holds for any theory T and any small ordinal α .*

Proof. (1), (2) and (3) follow immediately from Main Theorem B. (3) is also connected to Corollary 1.32.

(4) Suppose T is small. Let $n < \omega$, let a be some tuple of length n and let $Y = S_n(a)$. This is a countable Polish space. Thus every subset of Y is G_δ , in particular the set

$$Q = \{q \in S_n(a) \mid \forall b \models q \ (b \equiv_L^n a)\}.$$

(which can also be defined with \exists). Let M be any countable model containing a . Then the restriction map $\pi : S(M) \rightarrow S(a)$ is continuous. Thus, $\pi^{-1}(Q)$ is also G_δ . But it is exactly the Lascar strong type of a in $S(M)$. By (3), this class is d -bounded, but then by Remark 1.21 $\equiv_L^n \upharpoonright [a]_{\equiv_{KP}^n}$ is trivial and hence $\equiv_{KP}^n = \equiv_L^n$. \square

2. DESCRIPTIVE SET THEORETIC LEMMAS

2.1. Polish spaces. Given a group Γ of homeomorphisms of a topological space X , we use E_Γ^X to denote the corresponding orbit equivalence relation. Although the following fact can be seen as a consequence of the proof of [BK96, Theorem 3.4.5], for the sake of completeness we provide a proof.

Theorem 2.1. *Suppose that X is a perfect Polish space, Γ is a group of homeomorphisms of X with a dense orbit, and $R \subseteq X \times X$ is a meager set. Then there is a continuous, injective homomorphism $\phi : 2^\omega \rightarrow X$ from $(\mathbb{E}_0, \sim_{\mathbb{E}_0})$ to (E_Γ^X, \sim_R) .*

Proof. We use 1_Γ to denote the identity element of Γ . Given a natural number n and a sequence $\langle \gamma_i \mid i < n \rangle$ of elements of Γ , we use $\prod_{i < n} \gamma_i$ to denote 1_Γ when $n = 0$, and the product $\gamma_0 \cdots \gamma_{n-1}$ when $n > 0$. When $\langle \gamma_i \mid i < n \rangle$ is constant with value γ , we also use γ^n to denote $\prod_{i < n} \gamma_i$.

As X is perfect, the set of pairs of distinct points of X is comeager, so there is a decreasing sequence $\langle U_n \mid n \in \mathbb{N} \rangle$ of dense, irreflexive, open, symmetric subsets of $X \times X$ whose intersection is disjoint from R . We will recursively construct group elements $\gamma_n \in \Gamma$, with which we associate the products $\gamma_s = \prod_{i < n} \gamma_i^{s(i)}$, for all $n \in \mathbb{N}$ and $s \in 2^n$. We will simultaneously construct points $x_n \in X$ and open neighborhoods X_n of x_n with the following properties:

- (1) $\overline{X_{n+1}} \subseteq X_n \cap (\gamma_n^{-1} \cdot X_n)$.
- (2) $\forall s \in 2^{n+1} \text{ diam}(\gamma_s \cdot X_{n+1}) \leq 1/n$.
- (3) $\forall s, t \in 2^{n+1} (s(n) \neq t(n) \Rightarrow (\gamma_s \cdot X_{n+1}) \times (\gamma_t \cdot X_{n+1}) \subseteq U_n)$.

We begin by fixing an arbitrary point $x_0 \in X$ and setting $X_0 = X$.

Suppose now that $n \in \mathbb{N}$ and we have already found $\langle \gamma_m \mid m < n \rangle$, x_n , and X_n . The fact that Γ consists of homeomorphisms then ensures that the set

$$V_n = \bigcap \left\{ (\gamma_s \times \gamma_t)^{-1}(U_n) \mid (s, t) \in 2^n \times 2^n \right\}$$

is dense and open, so the fact that Γ has a dense orbit yields $\gamma_n \in \Gamma$ and $x_{n+1} \in X_n \cap (\gamma_n^{-1} \cdot X_n)$ for which $(x_{n+1}, \gamma_n \cdot x_{n+1}) \in V_n$. As Γ consists of homeomorphisms and U_n is symmetric, there is an open neighborhood X_{n+1} of x_{n+1} satisfying conditions (1) – (3). This completes the recursive construction.

Conditions (1) and (2) ensure that we obtain a continuous function $\phi : 2^\omega \rightarrow X$ by setting $\phi(c) = \lim_{n \rightarrow \infty} \gamma_{c \upharpoonright n} \cdot x_n$. To see that ϕ is a homomorphism from \mathbb{E}_0 to E_Γ^X , it is sufficient to observe that if $k \in \mathbb{N}$, $s \in 2^k$, and $y \in 2^\omega$, then

$$\phi(s \smallfrown y) = \lim_{n \rightarrow \infty} \gamma_{s \smallfrown y \upharpoonright n} \cdot x_n = \lim_{n \rightarrow \infty} \gamma_s \gamma_{(0)^k \smallfrown y \upharpoonright n} \cdot x_n = \gamma_s \cdot \phi\left((0)^k \smallfrown y\right).$$

Observe now that if $y, z \in 2^\omega$ and $y(n) \neq z(n)$, then conditions (1) and (3) ensure that $(\phi(y), \phi(z)) \in (\gamma_{y \upharpoonright (n+1)} \cdot X_{n+1}) \times (\gamma_{z \upharpoonright (n+1)} \cdot X_{n+1}) \subseteq U_n$, so the irreflexivity of U_n yields the injectivity of ϕ , and the fact that $\langle U_n \mid n \in \mathbb{N} \rangle$ is a decreasing sequence whose intersection is disjoint from R ensures that ϕ is a homomorphism from $\sim \mathbb{E}_0$ to $\sim R$. \square

Given $R \subseteq X \times X$ and $x \in X$, define $R_x = \{y \in X \mid x R y\}$.

Theorem 2.2. *Suppose that X is a Polish space, $\langle R_n \mid n \in \mathbb{N} \rangle$ is a sequence of F_σ subsets of $X \times X$, Γ is a group of homeomorphisms of X , and $\mathcal{O} \subseteq X$ is an orbit of Γ with the property that for all $n \in \mathbb{N}$ and open sets $U \subseteq X$ intersecting \mathcal{O} , there are distinct $x, y \in \mathcal{O} \cap U$ with $\mathcal{O} \cap (R_n)_x \cap (R_n)_y = \emptyset$. Then there is a continuous, injective homomorphism $\phi : 2^\omega \rightarrow \overline{\mathcal{O}}$ from $(\mathbb{E}_0, \sim \mathbb{E}_0)$ to $(E_\Gamma^X, \sim \bigcup \{R_n \mid n \in \mathbb{N}\})$.*

Proof. In light of Theorem 2.1, it is sufficient to show that $\overline{\mathcal{O}}$ is perfect and $\bigcup \{R_n \mid n \in \mathbb{N}\} \upharpoonright \overline{\mathcal{O}}$ is meager. For the former, observe that if $U \subseteq X$ is an open set intersecting $\overline{\mathcal{O}}$, then it intersects \mathcal{O} , so there are distinct $x, y \in \mathcal{O} \cap U$. For the latter, it is sufficient to check that each of the sets $\sim R_n \upharpoonright \overline{\mathcal{O}}$ is dense. Towards this end, suppose that $U, V \subseteq X$ are open sets intersecting $\overline{\mathcal{O}}$, and therefore \mathcal{O} . Then there exist $x, y \in \mathcal{O} \cap U$ with $\mathcal{O} \cap (R_n)_x \cap (R_n)_y = \emptyset$, as well as $z \in \mathcal{O} \cap V$, so $\neg x R_n z$ or $\neg y R_n z$, thus $\sim R_n \cap \mathcal{O} \cap (U \times V) \neq \emptyset$. \square

We are going to apply this in our context via:

Corollary 2.3. *Let T be a countable first order theory, let α be a countable ordinal and M a countable model. Let Y be a Polish subspace of $S_\alpha(M)$ that is closed under $\equiv_L^{\alpha, M}$. Suppose that there is some $\mathbf{x} \in Y$ such that for every open set $U \ni \mathbf{x}$ and for all $N \in \mathbb{N}$, there exist some $\sigma \in \text{Aut } f_L(\mathfrak{C})$ such that:*

- (1) *The automorphism σ^* that σ induces on $S_\alpha(M)$ fixes Y setwise.*
- (2) *$\sigma^*(\mathbf{x}) \in U$ and $N < d_\alpha(\sigma^*(\mathbf{x}), \mathbf{x})$ (see Definition 1.38).*

Then there is a continuous, injective homomorphism $\phi : 2^\omega \rightarrow Y$ from $(\mathbb{E}_0, \sim \mathbb{E}_0)$ to $(\equiv_L^{\alpha, M}, \sim \equiv_L^{\alpha, M})$. In particular, $\equiv_L^{\alpha, M} \upharpoonright Y$ is not smooth.

Proof. For $n < \omega$, let R_n be the closed set $\{(p, q) \in Y \times Y \mid d_\alpha(p, q) \leq n\}$. Let Γ be the group of homeomorphisms of Y which are induced by automorphisms in $\text{Aut } f_L(\mathfrak{C})$ which fix Y setwise. Let \mathcal{O} be the orbit of \mathbf{x} under Γ .

Let $n \in \mathbb{N}$ and let W be an open set which intersects \mathcal{O} . Then for some $\gamma \in \Gamma$, $\gamma(\mathbf{x}) \in W$. Let $U = \gamma^{-1}(W)$. Then for some $\sigma \in \text{Aut } f_L(\mathfrak{C})$, $\sigma^* \in \Gamma$, $\sigma^*(\mathbf{x}) \in U$ and $2n < d_\alpha(\sigma^*(\mathbf{x}), \mathbf{x})$. Let $x = \gamma(\mathbf{x})$ and $y = \gamma\sigma^*(\mathbf{x})$. So $d_\alpha(x, y) = d_\alpha(\mathbf{x}, \sigma^*(\mathbf{x})) > 2n$ and so x and y are distinct and $\mathcal{O} \cap (R_n)_x \cap (R_n)_y = \emptyset$. \square

2.2. Choquet spaces. In order to prove Main Theorem B, we have to work over a model M of possibly uncountable size, hence $S(M)$ is no longer a Polish space. The idea is to mimic the proof of Main Theorem A, i.e., construct step-by-step an embedding of \mathbb{E}_0 . In the countable case we use completeness at the limit stage, but here we use the winning strategy in the (strong) Choquet game.

The main observation is that Theorem 2.1 has a natural analog in the Choquet context:

Theorem 2.4. *Suppose that X is regular topological space, Γ is a group of homeomorphisms of X and \mathcal{O} an orbit of Γ such that X is Choquet over \mathcal{O} . Suppose that for $n < \omega$, $V_n \subseteq X \times X$ is a G_δ subset such that $V_n \upharpoonright \overline{\mathcal{O}} \times \overline{\mathcal{O}}$ is dense. Then there is a map $\phi : 2^\omega \rightarrow \mathcal{P}(X)$ such that for every $y, z \in 2^\omega$:*

- $\phi(y)$ is a non-empty closed G_δ subset of X .
- If $z \mathbb{E}_0 y$ then there is some $\gamma \in \Gamma$ such that $\gamma \cdot \phi(z) = \phi(y)$.
- If $\sim z \mathbb{E}_0 y$ then $(\phi(y) \times \phi(z)) \subseteq \bigcap_{n < \omega} V_n$.

Proof. The proof follows along the lines of the proof of Theorem 2.1. The main difference is that in condition (2) in the construction, instead of controlling the diameter of the open sets, one has to refine them so that they obey the winning strategy of player B in the suitable Choquet game over \mathcal{O} . \square

From this we get the following analog of 2.2:

Theorem 2.5. *Suppose that X is a topological space, $\langle R_n \mid n \in \mathbb{N} \rangle$ is a sequence of F_σ subsets of $X \times X$, Γ is a group of homeomorphisms of X , and $\mathcal{O} \subseteq X$ is an orbit of Γ with the property that for all $n \in \mathbb{N}$ and open sets $U \subseteq X$ intersecting \mathcal{O} , there are distinct $x, y \in \mathcal{O} \cap U$ with $\mathcal{O} \cap (R_n)_x \cap (R_n)_y = \emptyset$. If X is strong Choquet over \mathcal{O} then the conclusion of Theorem 2.4 holds with $V_n = \sim R_n$.*

And:

Corollary 2.6. *Let T be any first order theory with language L , let α be any ordinal and M a model. Let Y be a subspace of $S_\alpha(M)$ that is closed under $\equiv_L^{\alpha, M}$. Suppose that there is*

- (1) Some $\mathbf{x} \in Y$.
- (2) A countable sub-language L' of L , a countable set $M' \prec M \upharpoonright L'$ and a countable sub-tuple of the first α variables which for simplicity we will assume to be the initial segment of length β .
- (3) A countable subgroup $\Sigma \leq \text{Aut } f_L(\mathfrak{C})$ of automorphisms that fix M' and M setwise.

Such that:

- (1) With the topology induced on Y by L' , M' , and β (the one generated by formulas in $L'_\beta(M')$), Y is strong Choquet over $\Sigma \cdot \mathbf{x}$.
- (2) For every open set $U \ni \mathbf{x}$ in the induced topology and for all $N \in \mathbb{N}$, there exist some $\sigma \in \Sigma$ such that $\sigma^*(\mathbf{x}) \in U$ and, letting $\mathbf{x}' = \mathbf{x} \upharpoonright L_\beta(M')$, $N < d'_\beta(\sigma^*(\mathbf{x}'), \mathbf{x}')$ (d'_β is the Lascar metric of the language L').

Then there is a map $\phi : 2^\omega \rightarrow \mathcal{P}(Y)$ such that for every $y, z \in 2^\omega$:

- $\phi(y)$ is a non-empty closed G_δ subset of Y .
- If $z \mathbb{E}_0 y$ then there is a some $\gamma \in \Gamma$ such that $\gamma \cdot \phi(z) = \phi(y)$.
- If $\sim z \mathbb{E}_0 y$ then $(\phi(y) \times \phi(z)) \cap \equiv_L^{\alpha, M} = \emptyset$.

Proof. Similar to 2.3. Note that if $p, q \in S_\alpha(M)$ and $p \upharpoonright L'_\beta(M'), q \upharpoonright L'_\beta(M')$ are not $\equiv_L^{\beta, M'}$ equivalent, then p, q are not $\equiv_L^{\alpha, M}$ -equivalent. \square

The following lemma will not be used directly, but its proof will give insight into the proof of Main theorem B.

Lemma 2.7. *Suppose (X, τ) is a Choquet space with topology τ . Let $B \subseteq \mathcal{P}(X)$ be a base for τ , and assume it is closed under finite intersections. Let $B_0 \subseteq B$. Then there exists $B_0 \subseteq B_1 \subseteq B$ such that $|B_1| \leq |B_0| + \aleph_0$ and (X, τ_{B_1}) is Choquet, where τ_{B_1} is the topology generated by B_1 .*

Proof. Let St be a winning strategy for player B in the Choquet game of (X, τ) . Let $s = \langle U_i \mid i \leq n \rangle$ be a finite sequence of elements of B_0 . Suppose s consists of a legal $n+1$ -play of player A, where player B plays his moves according to St for $i < n$. Let V_s be a nonempty basic open set contained in player B's play in the n 'th round of his move according to St . Let B_0^1 be the closure under finite intersections of $B_0 \cup \{V_s \mid s \in B_0^{<\omega}\}$. This is a subset of B . Repeat this recursively to construct B_0^n for $n < \omega$, and let $B_1 = \bigcup_{n < \omega} B_0^n$. Then B_1 satisfies the cardinality demand. Let us see that (X, τ_{B_1}) is Choquet. For this we must describe a winning strategy for player B.

So suppose $\langle (U_i, V_i) \mid i < n \rangle$ is a legal play of the Choquet game in τ_{B_1} (where U_i is played by player A and V_i is played by player B), and player A chooses U_n . Suppose that:

- There are basic open sets $U'_i \in B_1$ and open sets $V'_i \in \tau$ for $i < n$ such that $\langle (U'_i, V'_i) \mid i < n \rangle$ is a play of the Choquet game compatible with St , $U'_i \subseteq U_i$ and $V_i \subseteq V'_i$.

Let $U'_n \in B_1$ be such that $U'_n \subseteq U_n$. There is some $m < \omega$ such that $U'_i \subseteq B_0^m$ for all $i < n$ and let $s = \langle U'_i \mid i \leq n \rangle$. By construction of B_0^{m+1} , $V_s \in B_1$ so let player B play V_s .

If this does not hold, let player B play any set.

Now it is easy to see that if player B plays according to this strategy, then he will win the game. \square

3. THE SMALL CASE

Here we prove Main Theorem A under the assumption that a consequence of smallness holds, namely that the conclusion of Fact 1.1 (4) holds. This result is superseded by Theorem 4.12 in the next section, and the reader may skip it if desired.

Assume that T is a complete theory in a countable language L and that \mathfrak{C} is a monster model for T .

Claim 3.1. Suppose that A is a countable set and that $\{\sigma_i \mid i < \omega\}$ is a set of automorphisms of \mathfrak{C} . Then there is a countable model $N \supseteq A$ such that $\sigma_i \upharpoonright N$ is an automorphism of N for all $i < \omega$.

Proof. Let M_0 be some model containing A , and for $n > 0$, let M_n be a countable model containing $\bigcup_{j \in \mathbb{Z}, i < \omega} \sigma_i^{(j)}(M_{n-1})$. Let $N = \bigcup_{n < \omega} M_n$. \square

Definition 3.2. Call a countable model M of T *nice* if the following conditions hold:

- (1) For every pair of finite tuples $a, b \in M^k$, if $a \equiv_L^k b$ then there is a Lascar strong automorphism σ of M (i.e., $\sigma \in \text{Aut } f_L(\mathfrak{C}) \cap \text{Aut}(M)$) that maps a to b . Moreover, σ has minimal bound (see Fact 1.17 (2)) among all automorphisms in $\text{Aut } f_L(\mathfrak{C})$ that map a to b .
- (2) For every finite tuple $a \in M^k$, and every $n < \omega$, if there are $c_1, c_2 \in \mathfrak{C}^k$ such that $c_1 \equiv_L^k a \equiv_L^k c_2$ and $d_k(c_1, c_2) > n$, then there are such c_1, c_2 in M^k .
- (3) For all finite tuples $a, b \in M^k$ and $a' \in M^{k'}$ and every $n < \omega$, if there is some b' such that $d_{k+k'}(a \smallfrown a', b \smallfrown b') \leq n$, then there is some such $b' \in M^{k'}$.

Lemma 3.3. *Nice models exist. Moreover, for every countable set A , there is a nice model M that contains it.*

Proof. Let M_0 be any countable model containing A . Recursively choose M_{n+1} to satisfy (1)–(3) relative to M_n (using Claim 3.1) and set $M = \bigcup_{n < \omega} M_n$. \square

Fix a countable ordinal α and a pseudo G_δ set Y . Assume that:

- Assumption A.**
- (1) α is infinite.
 - (2) The Lascar strong type of every finite sub-tuple of a tuple from Y is d -bounded.

Remark 3.4. By Fact 1.1 (4), if T is small, then for finite tuples, $\equiv_{KP} = \equiv_L$, so this assumption is satisfied when α is infinite if T is small, and Corollary 1.46 is trivial for finite α (given Fact 1.1 (1)).

Theorem 3.5. *Main Theorem A holds under Assumption A.*

Namely, suppose α and Y are as above and for some $\bar{a} \in Y$, $[\bar{a}]_{\equiv_L^\alpha}$ is not d -bounded. Then $\equiv_L^\alpha \upharpoonright Y$ is non-smooth.

Proof. Choose a nice model M (by Lemma 3.3) that contains \bar{a} .

Let $\mathbf{x} = p = \text{tp}(\bar{a}/M)$. We shall show that corollary 2.3 applies with Y there being Y_M (see Proposition 1.41).

Suppose U is some open set containing p , and N is some number. In general, U has the form $[\varphi]$ for some $\varphi \in L_\alpha(M)$. But in our case, since $M \models \varphi(\bar{a})$, we may replace U with a smaller open neighborhood of p defined by a formula of the form $x = c$, where x is the tuple of first k variables and $c = \bar{a} \upharpoonright k$.

Let B be a bound on the diameter of $[c]_{\equiv_L^k}$. Since the class of \bar{a} is not of bounded diameter, by compactness there must be some finite extension of c to a longer sub-tuple $c \frown c' = \bar{a} \upharpoonright (k + k')$ such that the $\equiv_L^{k+k'}$ -class of $c \frown c'$ has diameter greater than $2N + 4B$.

There are two tuples $f_1 \frown f'_1$ and $f_2 \frown f'_2$ in \mathfrak{C} and $[c \frown c']_{\equiv_L^{k+k'}}$ such that $d_{k+k'}(f_1 \frown f'_1, f_2 \frown f'_2) > 2N + 2B$. Since M is nice, we may assume that these tuples are in M .

By choice of B , niceness of M and Remark 1.18, there are c'' and c''' in M such that

$$d_{k+k'}(f_1 \frown f'_1, c \frown c''), d_{k+k'}(f_2 \frown f'_2, c \frown c''') \leq 2B.$$

So $d_{k+k'}(c \frown c'', c \frown c''') > 2N$. It follows that for one of c'', c''' , say c'' , $d_{k+k'}(c \frown c', c \frown c'') > N$ (but $c \frown c' \equiv_L^{k+k'} c \frown c''$).

Let σ be a Lascar strong automorphism of M that maps $c \frown c'$ to $c \frown c''$. Since σ fixes c , $q = \sigma^*(p) \in U$. But q is realized by a tuple that contains $c \frown c''$, and hence $d_\alpha(q, p) > N$. \square

4. THE COUNTABLE CASE

Assume that α is a countable ordinal, T is a complete theory in a countable language L and \mathfrak{C} is a monster model for T .

Definition 4.1. For a formula $\alpha(x, a)$ over a tuple a and an automorphism σ , $\sigma(\alpha) = \alpha(x, \sigma(a))$.

Definition 4.2. Suppose $C \subseteq \mathfrak{C}^\alpha$ is an \equiv_L^α -class. A formula $\varphi \in L_\alpha(\mathfrak{C})$ is said to be C -generic if finitely many translates of it under $\text{Aut } f_L(\mathfrak{C})$ cover C . The formula φ is said to be C -weakly generic if there is a non- C -generic formula $\psi \in L_\alpha(\mathfrak{C})$ such that $\varphi \vee \psi$ is C -generic. A partial $p \subseteq L_\alpha(\mathfrak{C})$ is said to be C -generic (C -weakly generic) if all its formulas are.

Claim 4.3. The formulas which are not C -weakly generic form an ideal.

Proof. Suppose φ_1, φ_2 are not C -weakly generic and we have to show that $\varphi_1 \vee \varphi_2$ is also not C -weakly generic. If not, there is some non- C -generic ψ such that $\varphi_1 \vee \varphi_2 \vee \psi$ is C -generic. But $\varphi_2 \vee \psi$ is not C -generic (since φ_2 is not C -weakly generic), so we get a contradiction. \square

By $\varphi \vdash_C \psi$ we mean that for every $a \in C$, if $\mathfrak{C} \models \varphi(a)$ then $\mathfrak{C} \models \psi(a)$.

Remark 4.4. If $\varphi \vdash_C \psi$ and $\sigma \in \text{Aut } f_L(\mathfrak{C})$ then $\sigma(\varphi) \vdash_C \sigma(\psi)$, so if φ is (weakly) generic, then so is ψ .

Definition 4.5. Suppose p is a weakly generic (partial) type over \mathfrak{C} . Suppose furthermore that p is closed under conjunctions. Say that it is *C-proper* if there is a non- C -generic formula ψ such that for all $\varphi \in p$, $\varphi \vee \psi$ is C -generic. In general, p is C -proper when its closure under finite conjunctions is.

Fix an \equiv_L^α -class C . When we write “(weakly) generic” and “proper”, we mean “ C -(weakly) generic” and “ C -proper”.

Example 4.6. If p is generic, then it is proper.

An easy and well known combinatorial lemma is the following:

Lemma 4.7. *If $(P, <)$ is a directed order, $k < \omega$ and $f : P \rightarrow k$ is some function, then there is a cofinal f -homogeneous set $P_0 \subseteq P$: there is some $i < k$ such that $f^{-1}(i)$ is cofinal.*

Proof. Suppose not. So for each $i < k$, the $f^{-1}(i)$ is not cofinal, for some $p_i \in P$, for no $q \geq p_i$, $f(q) = i$. Let p be $\geq p_i$ for every $i < k$. Then $p \geq p_{f(p)}$ — contradiction. \square

Lemma 4.8. *Suppose $p \subseteq L_\alpha(\mathfrak{C})$ is a partial proper type as witnessed by ψ . Suppose that $\bigvee_{i < n} \varphi_i \vee \psi'$ covers C and that $\psi' \vee \psi$ is non-generic. Then for some $i < n$, $p \cup \{\varphi_i\}$ is proper.*

Proof. We may assume that p is closed under conjunctions. For each formula $\zeta \in p$, by assumption we have:

$$\zeta \vee \psi \vdash_C \bigvee_{i < n} (\varphi_i \wedge \zeta) \vee \psi' \vee \psi.$$

So by Remark 4.4, the right hand side is generic.

For each $\zeta \in p$ and $k < n$, let $\zeta_k = \bigvee_{k \leq i < n} (\varphi_i \wedge \zeta) \vee \psi' \vee \psi$. Let $k_\zeta < n$ be maximal such that ζ_k is generic (must exist since ζ_0 is generic), so ζ_{k+1} is non-generic. By Lemma 4.7, for some $k < n$, the set $\{\zeta \mid k_\zeta = k\}$ is cofinal in the order $\zeta_1 > \zeta_2 \Leftrightarrow \zeta_1 \vdash \zeta_2$. Fix some $\chi \in p$ such that $k_\chi = k$. We will show that $p \cup \{\varphi_k\}$ is proper, as witnessed by χ_{k+1} .

Suppose $\zeta \in p$. Let $\zeta' = \zeta \wedge \chi$, and $\zeta'' \vdash \zeta'$ be such that $k_{\zeta''} = k$. Then $(\zeta'' \wedge \varphi_k) \vee \zeta''_{k+1}$ is generic. Since $\zeta'' \wedge \varphi_k \vdash \zeta \wedge \varphi_k$ and $\zeta''_{k+1} \vdash \chi_{k+1}$, $(\zeta \wedge \varphi_k) \vee \chi_{k+1}$ is also generic and we are done. \square

Lemma 4.9. *If $p \subseteq L_\alpha(\mathfrak{C})$ is a partial proper type, then for every formula $\varphi \in L_\alpha(\mathfrak{C})$, either $p \cup \{\varphi\}$ is proper or $p \cup \{\neg\varphi\}$ is proper.*

Proof. Apply Lemma 4.8 with $n = 2$, $\varphi_0 = \varphi$, $\varphi_1 = \neg\varphi$ and $\psi' = \perp$ (i.e., $\forall x (x \neq x)$).

Note that if we do not care about properness but only about weak genericity, then this follows directly from Claim 4.3. \square

Proposition 4.10. *Suppose that $p \subseteq L_\alpha(\mathfrak{C})$ is a partial proper type, and $\varphi \in p$. Then there are $\sigma_0, \dots, \sigma_{n-1} \in \text{Aut } f_L(\mathfrak{C})$ such that for every $\sigma \in \text{Aut } f_L(\mathfrak{C})$, there exists some $i < n$ such that $p \cup \{\sigma(\sigma_i(\varphi))\}$ is proper.*

Proof. Since p is proper, there is some non-generic formula $\psi(x)$ that witnesses it. In particular, there is some $n < \omega$ and some $\sigma_0, \dots, \sigma_{n-1} \in \text{Aut } f_L(\mathfrak{C})$ such that $\bigvee_{i < n} \sigma_i(\varphi \vee \psi)$ covers C .

Suppose that $\sigma \in \text{Aut } f_L(\mathfrak{C})$. Then $\sigma(\bigvee_{i < n} \sigma_i(\varphi \vee \psi)) = \bigvee_{i < n} \sigma(\sigma_i(\varphi \vee \psi))$ also covers C . Since ψ is non-generic, $\psi' = \bigvee_{j < n} \sigma(\sigma_j(\psi))$ is also non-generic and so is $\psi' \vee \psi$.

Now we can apply Lemma Lemma 4.8. \square

Proposition 4.11. *Let $a \in C$. Consider the partial type $q(x, y) = d_\alpha(x, y) \leq 1$. Then $q(x, a) \subseteq L_\alpha(a)$ is generic and hence proper.*

Proof. Suppose $\varphi(x, a)$ is a formula in $q(x, a)$. Suppose φ is non-generic. This means that for every n Lascar strong conjugates a_0, \dots, a_{n-1} of a , there is some $a' \in C$ (so another Lascar conjugate of a) such that $\neg\varphi(a', a_i)$ holds for all $i < n$. Thus we can construct an infinite sequence $\langle a_i \mid i < \omega \rangle$ of Lascar conjugates of a such that for every $i < \omega$, $\neg\varphi(a_i, a_j)$ holds for all $j < i$.

By Fact 1.10 (1), there is an indiscernible sequence $\langle b_i \mid i < \omega \rangle$ such that for all $j < i < \omega$, $\neg\varphi(b_i, b_j)$ holds. But this is a contradiction because by definition $d_\alpha(b_1, b_0) \leq 1$. \square

Theorem 4.12. *Main Theorem A holds:*

Suppose T is a complete countable first-order theory, α a countable ordinal, and suppose Y is a pseudo G_δ subset of \mathfrak{C}^α which is closed under \equiv_L^α . If for some $a \in Y$, $[a]_{\equiv_L^\alpha}$ is not d -bounded, then $\equiv_L^\alpha \upharpoonright Y$ is non-smooth.

Proof. Let $C = [a]_{\equiv_L^\alpha}$. For what follows when we write proper, we mean C -proper.

We want to apply Corollary 2.3 with Y there being Y_M (See Proposition 1.41) for some countable model M . Hence we will construct a pair (M, p) such that M is a countable model of T and $\mathbf{x} = p \in Y_M$ satisfies the condition in Corollary 2.3. Translating, this means that for every formula $\varphi \in p$, and every $N < \omega$, there exists some Lascar strong automorphism σ such that $\sigma(M) = M$, $\varphi \in \sigma(p)$ and $d_\alpha(\sigma(p), p) > N$.

Let $q(x) = d_\alpha(x, a) \leq 1$ (as in Proposition 4.11). We construct a sequence $\langle \sigma_i, p_i, M_i \mid i < \omega \rangle$ such that:

- (1) M_i is a finite set for all $i < \omega$.
- (2) $M_i \subseteq M_{i+1}$ and $p_i \subseteq p_{i+1}$ for all $i < \omega$.
- (3) For all $i < \omega$, p_i is a finite type over M_i such that $p_i \cup q$ is proper.
- (4) For all $i < \omega$, σ_i is a Lascar strong automorphism.
- (5) For every $i < \omega$ and formula of $\varphi \in L_1(M_i)$, if φ is not empty then for some $i < j < \omega$ there is some $c \in M_j$ such that $\varphi(c)$ holds.

- (6) For every $i < \omega$ and $n < \omega$ there exists some $i < j < \omega$ such that M_j contains $\bigcup_{-n < l < n, i' < i} \sigma_{i'}^{(l)}(M_i)$.
- (7) For every $i < \omega$ and formula $\varphi \in L_\alpha(M_i)$, there is some $i < j < \omega$ such that p_j contains either φ or $\neg\varphi$.
- (8) For every $i < \omega$, $N < \omega$ and $\varphi \in p_i$ there are some $i < j < j' < \omega$ such that $d_\alpha(\sigma_j(a), a) > N$ and $\sigma_j^{-1}(\varphi) \in p_{j'}$.

If we succeed, then let $M = \bigcup M_i$, $p = \bigcup p_i$. M is a model by (5), and $p \in S(M)$ and even belongs to Y_M by (7) and (3).

(3), (6) and (8) imply that (M, p) satisfy the required condition: for every formula $\varphi(x) \in p$ (i.e., $p \in [\varphi]$) and $N < \omega$, there is some σ_j as in (8). By (3), $d_\alpha(c, a) \leq 3$ for any $c \models p$ (because there exists some $c' \models p \cup q$, and $d_\alpha(c, c') \leq 2$). So $d_\alpha(c, \sigma_j(a)) > N - 3$. By (6) $\sigma_j(M) = M$, and so $d_\alpha(p, \sigma_j^*(p))$ is well defined and $> N - 6$ (as for any $c \models \sigma_j^*(p)$, $d_\alpha(c, \sigma_j(a)) \leq 3$). Finally, $\varphi \in \sigma_j^*(p)$.

The construction:

Let M_0 and p_0 be \emptyset . Note that condition (3) holds by Proposition 4.11.

Now we partition the work so that can satisfy all conditions. In each stage we take care of one of (5)–(8).

(5) and (6) are easy (just add some elements to M_i). (7) can be achieved by Lemma 4.9.

For (8) we need some argument. So suppose we are in stage $i + 1$ of the construction and we deal with (8), i.e., we are given $N < \omega$ and $\varphi \in p_i$. By Proposition 4.10, there are $\tau_0, \dots, \tau_{n-1} \in \text{Aut } f_L(\mathfrak{C})$ such that for every $\sigma \in \text{Aut } f_L(\mathfrak{C})$, there exists some $j < n$ such that $q \cup p_i \cup \{\sigma(\tau_j(\varphi))\}$ is proper. There is some bound k on τ_j for all $j < n$. Let $\sigma \in \text{Aut } f_L(\mathfrak{C})$ be such that $d_\alpha(a, \sigma(a)) > N + k$. By the triangle inequality, $d_\alpha(a, \sigma(\tau_j(a))) > N$ for all $j < n$. For some $j < n$, $q \cup p_i \cup \{\sigma(\tau_j(\varphi))\}$ is proper, so let $\sigma_{i+1} = (\sigma \circ \tau_j)^{-1}$ (note that $d_\alpha(a, (\sigma \circ \tau_j)(a)) = d_\alpha(a, (\sigma \circ \tau_j)^{-1}(a))$) and $p_{i+1} = p_i \cup \{\sigma(\tau_j(\varphi))\}$ and continue. \square

5. THE GENERAL CASE

Here we adapt our techniques to the case where the language is not necessarily countable.

Theorem 5.1. *Main theorem B holds:*

Suppose T is a complete first-order theory, α a small ordinal. Suppose $Y \subseteq \mathfrak{C}^\alpha$ is closed under \equiv_L^α and for some $a \in Y$, $[a]_{\equiv_L^\alpha}$ is not d -bounded. Suppose Y is pseudo strong Choquet. Then there is a model M of size $|T| + |\alpha|$ and a function $\phi : 2^\omega \rightarrow \mathcal{P}(Y_M)$ such that for every $y, z \in 2^\omega$:

- $\phi(y)$ is a non-empty closed G_δ subset of Y_M .
- If $z \mathbb{E}_0 y$ then there is a some $\gamma \in \Gamma$ such that $\gamma \cdot \phi(z) = \phi(y)$.
- If $\sim z \mathbb{E}_0 y$ then $(\phi(y) \times \phi(z)) \cap \equiv_L^{\alpha, M} = \emptyset$.

In particular, $|Y/\equiv_L^\alpha| \geq 2^{\aleph_0}$.

Proof. The idea is to simultaneously construct a countable language L' , a countable model M' , a countable sub-tuple of the first α variables, an L' -type over M' in these variables and a countable group of Lascar strong automorphisms so that we can apply Corollary 2.6. Eventually, \mathbf{x} will be any completion of the L' -type over M' to a complete L -type over M .

So we will need a more elaborate argument than the one used in Theorem 4.12 that will also use the proof of Lemma 2.7 (but not the lemma itself). That is, we try to construct the winning strategy along with the model and language.

Let $C = [a]_{\equiv_L^\alpha}$. For what follows when we write proper, we mean C -proper. Fix a countable set S of Lascar strong automorphisms that witness that C is not d -bounded, i.e., such that for all $N > 0$, there is some $\sigma \in S$ such that $d_\alpha(a, \sigma(a)) > N$.

Let M be a model of T of size $|T| + |\alpha|$ that contains a such that every $\sigma \in S$ fixes M setwise and for every generic formula over M , there are Lascar strong automorphisms that witness it which fix M setwise. Such a model can be constructed as in Claim 3.1. Let Γ be the group of Lascar strong automorphisms that fix M setwise. Let St be a strategy for player B that witnesses that Y_M is strong Choquet.

We construct:

- A countable sub-language $L' \subseteq L$.
- A countable model $M' \prec M \upharpoonright L'$.
- A countable sub-tuple x' of the first α variables. For notational simplicity we will assume that x' is the first β variables for a countable ordinal β .
- A complete L' -type p over M_0 in x' which is consistent with the type $q(x) = d_\alpha(x, a) \leq 1$.
- A countable subgroup $\Sigma \subseteq \Gamma$ of automorphism that fix M' and M setwise.
- A countable set Q of complete types in $S_\alpha(M)$ contained in Y_M .

Such that:

- (1) For every formula $\varphi \in p$ and natural number N , there is an automorphism $\sigma \in \Sigma$ such that $\sigma^{-1}(\varphi) \in p$ and $d'_\beta(\sigma^*(p), p) > N$ where d' is the Lascar metric when restricted to L' .
- (2) For every $\sigma_0, \dots, \sigma_n \in \Sigma$, $r_0, \dots, r_{n-1} \in Q$ and every sequence of $L'_\beta(M')$ formulas $\langle (\varphi_i, \psi_i) \mid i < n \rangle$ and a formula φ_n such that:
 - (a) $\varphi_{i+1} \vdash \psi_i \vdash \varphi_i$ for all $i < n$.
 - (b) $\psi_i \in \sigma_i^*(p)$ (in other words, $\sigma_i^*(p)$ is in the open set $[\psi_i]$).
 - (c) r_i is a complete extension of $\{\varphi_i\}$ consistent with $\sigma_i^*(q)$.
 - (d) $\varphi_n \in \sigma_n^*(p)$.

- (e) For each $i < n$, ψ_i is such that $[\psi_i] \cap Y_M$ is a basic open subset of player B's move according to St in the strong Choquet game where player A plays φ_i and r_i .

There is a type $r_n \in Q$ containing φ_n and a formula ψ_n in $\sigma_n^*(p) \cap r_n$ contained in φ_n which is a subset of player B's move according to St in the strong Choquet game described in (e) where in the n 'th move player A chooses φ_n and r_n .

For the construction we repeat the proof of Theorem 4.12 inside M . As there, we let $q = d(x, a) \leq 1$, and note that it is proper. The differences are:

- ★ We choose our automorphisms from Γ (this is no problem, since they all come from witnesses of genericity of certain formulas over M composed with an element from S by the proof of 4.10).
- ★ We have to take care of d' instead of d . So in (8) there we increase the language L' so that not only $d_\alpha(\sigma_j(a), a) > N$ is true in L , but it also true in L' . Similarly add some variables to the tuple of variables we construct so that this remains true when restricted to these variables.
- ★ We add a step to the construction that makes sure that the set of automorphisms is a group.
- ★ For (2), we add a step to the construction. We have to take care of every choice of $\sigma_0, \dots, \sigma_n \in \Sigma$, r_0, \dots, r_{n-1} , a formula φ_n and a sequence of $L'_\beta(M')$ formulas $\langle (\varphi_i, \psi_i) \mid i < n \rangle$ from the language and model constructed thus far that satisfy (a)–(e) above. We may assume that φ_n is the conjunction of σ_n applied to the current finite partial type we have. For every complete extension $r \in S_\alpha(M)$ of $\{\varphi_n\}$ consistent with $\sigma_n^*(q)$, there is some open set $r \in V_r \subseteq [\varphi_n] \cap Y_M$ that player B plays according to St in the strong Choquet game described in (e) where in the n 'th move player A chooses φ_n and r . Let ψ_r be a formula in $L_\alpha(M)$ that contains r (i.e., $\psi_r \in r$), $\psi_r \vdash \varphi_n$ and $[\psi_r] \cap Y_M$ is contained in V_r . It follows that $\{\varphi_n\} \cup \sigma_n^*(q) \vdash \bigvee \psi_r$. By compactness and by Lemma 4.9 for some r , $\sigma_n^*(q) \cup \{\varphi_n, \psi_r\}$ is proper. So we may add $\sigma_n^{-1}(\psi_r)$ to our partial type. Also we add the symbols appearing in ψ_r to the language and the variables appearing in it to the tuple of variables. Finally, add r to Q .

When the construction is done, it is easy to see that letting \mathbf{x} be any completion of the complete $L'_\beta(M')$ type constructed p , it satisfies all the demands of Corollary 2.6. For instance, we need to check that with the topology induced on Y_M by L' , M' , and β , Y is strong Choquet over $\Sigma \cdot \mathbf{x}$. The point is that if player A chooses some basic open set $[\varphi]$ containing $\sigma^*(\mathbf{x})$ for some $\sigma \in \Sigma$, then by construction there is some formula ψ in $\sigma^*(p)$ (so $[\psi]$ contains $\sigma^*(\mathbf{x})$) that is contained in $[\varphi]$ and some type $r_0 \in Q$ such that ψ is contained in player B's response to $([\varphi], r_0)$. So player B will now choose $[\psi] \cap Y_M$. So we simulate a game in Y_M in which player A chooses types from Q ,

and player B responds by choosing a basic open subset of what St says. Since St was a winning strategy, the intersection must be nonempty. \square

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